Quick Guide to Power Series

For Ryan Holben's Math 3D class, Winter 2015 at UC Irvine[1].

1 Introduction

The aim of this guide is to get you up and running with power series as quickly as possible, so that you can use them to solve differential equations. I have omitted a lot of background material about basic series, which we covered in class.

1.1 Basic forms

General power series are of the form

$$\sum_{n=k}^{\infty} a_n (x-a)^n.$$

This is a series centered at a. However, often our series will start at $n = 0$ and will be centered at $a = 0$, giving us a sum of the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

A specific kind of power series is called a Taylor series. The Taylor series centered at a for a function $f(x)$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Again, we will usually center our series at 0. This is sometimes called a Maclaurin series, and it looks like

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-a)^n$$

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2 Useful series

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}, \quad \cosh(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

$$\text{Geometric series, when } |r| < 1: \quad \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

$$p\text{-series converges for } p > 1: \quad \sum_{n=0}^{\infty} \frac{1}{n^p}$$

3 Radius of convergence

Let us see how to find the radius of convergence for a power series.

3.0.1 Example 1

$$\sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} (x-5)^n$$

First we do the ratio test:

$$\lim_{n\to\infty} \left| \frac{2^{n+1}(x-5)^{n+1}}{(n+1)^2 e^{n+1}} \frac{n^2 e^n}{2^n (x-5)^n} \right|$$

$$= \lim_{n\to\infty} \frac{2}{e} \left(\frac{n}{n+1} \right)^2 |x-5| = \frac{2}{e}|x-5|$$

The ratio test says the sum will converge if this limit is less than 1. Therefore to find out which x make our sum converge, we will write the inequality and solve for x.

$$\frac{2}{e}|x-5| < 1$$

$$|x-5| < \frac{e}{2}$$

This step tells us that our **radius of convergence** is $\frac{e}{2}$. Let's continue to find the interval of convergence:

$$-\frac{e}{2} < x - 5 < \frac{e}{2}$$

$$5 - \frac{e}{2} < x < \frac{e}{2} + 5$$

So the interval of convergence is from $5 - \frac{e}{2}$ to $5 + \frac{e}{2}$. But what about the endpoints? We must test them separately by plugging them in for x into the original sum and testing for convergence. First, let $x = 5 - \frac{e}{2}$. Our sum becomes

$$\sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \left(\left(5 - \frac{e}{2}\right) - 5 \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \left(\frac{-e}{2} \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \cdot \frac{(-1)^n e^n}{2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

This is the alternating sum version of the p-series for $p = 2$. Since that p-series converges and is the absolute version of our sum, we say that our series at $x = 5 - \frac{e}{2}$ is absolutely convergent, and therefore is convergent. Thus $5 - \frac{e}{2}$ is included in our interval of convergence. Next, let $x = 5 + \frac{e}{2}$.

$$\sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \left(\left(5 + \frac{e}{2}\right) - 5 \right)^n$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \left(\frac{e}{2} \right)^n = \sum_{n=0}^{\infty} \frac{2^n}{n^2 e^n} \cdot \frac{e^n}{2^n} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

This is a p-series with $p = 2$, so it converges. Thus $5 + \frac{e}{2}$ is included in our interval of convergence. So our series converges at both endpoints, so the **interval of convergence** is the closed interval $[5 - \frac{e}{2}, 5 + \frac{e}{2}]$.

3.0.2 Example 2

Let's do a faster example. Our series is

$$\sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n x^n$$

Do the ratio test:

$$\lim_{n \to \infty} \left| \frac{2^{n+1} x^{n+1}}{3^{n+1}} \frac{3^n}{2^n x^n} \right| = \lim_{n \to \infty} \frac{2}{3} |x| = \frac{2}{3} |x| < 1$$

So

$$|x| < \frac{3}{2}$$

$$-\frac{3}{2} < x < \frac{3}{2}$$

The radius of convergence is $\frac{3}{2}$. Test $x = -\frac{3}{2}$:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3} \right)^n \left(\frac{-3}{2} \right)^n = \sum_{n=0}^{\infty} (-1)^n \text{ diverges.}$$

And test $x = \frac{3}{2}$:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \left(\frac{3}{2}\right)^n = \sum_{n=0}^{\infty} 1 = \infty.$$

And so our series doesn't converge for either endpoint, so our interval of convergence is the open interval $(-\frac{3}{2}, \frac{3}{2})$.

4 Finding a Taylor series

4.0.3 Example: $f(x) = \dfrac{1}{x}$

Let's find the Taylor series for $f(x)$. Now power series are defined on intervals, but $\frac{1}{x}$ is not defined at 0. Therefore, any Taylor series we come up with for our function will end up being defined on an interval which does not contain 0. In your homework you do find the Taylor series for this function centered at 1. In this guide we'll center it at 2.

First, start taking derivatives so we can see if there is a pattern

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	x^{-1}	$\frac{1}{2}$
1	$(-1)x^{-2}$	$\frac{-1}{2^2}$
2	$(-1)(-2)x^{-3}$	$\frac{2}{2^3}$
$\vdots$	$\vdots$	$\vdots$

We see the pattern:

$$f^{(n)}(x) = \frac{(-1)^n n!}{x^n}, \text{ and so } f^{(n)}(2) = \frac{(-1)^n n!}{2^n}.$$

So the Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} \frac{1}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-2)^n$$

Finally let's quickly check the interval of convergence.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-2)^{n+1}}{2^{n+1}} \frac{2^n}{(-1)^n (x-2)^n} \right| = \lim_{n\to\infty} \frac{1}{2} |x-2| < 1$$

So $|x-2| < 2$ meaning our radius is 2. Our endpoints are 0 and 4.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(0-2)^n = \sum_{n=0}^{\infty} 1 = \infty \text{ and } \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(4-2)^n = \sum_{n=0}^{\infty} (-1)^n \text{ diverges,}$$

so neither endpoint is included. The interval of convergence is $(0, 4)$.

Remember that $f(x)$ is not defined at 0 and our series is centered at 2. Therefore $r = 2$ is the largest possible radius that works, and indeed that's the radius we found.

5 Manipulating sums

It is useful to do various operations to power series, like adding, scaling, differentiating, integrating, as well as reindexing them.

5.0.4 Example: $\frac{d}{dx}\textbf{sin}(x) = \textbf{cos}(x)$

We differentiate series term by term:

$$\frac{d}{dx}\sin(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{d}{dx}\left[\frac{(-1)^n}{(2n+1)!}x^{2n+1}\right] = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = \cos(x)$$

Whenever we take a derivative, we have to be careful about where our sum starts. Plug in the first index and make sure that taking the derivative of the first term won't make it go to 0. If it does, you need to omit that term from your series. In our case, if $n = 0$, the first term has $x^{2(0)+1} = x$, so taking its derivative won't eliminate it.

5.0.5 Example: $\frac{d}{dx}\textbf{cos}(x) = -\textbf{sin}(x)$

$$\frac{d}{dx}\cos(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{d}{dx}\left[\frac{(-1)^n}{(2n)!}x^{2n}\right] = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}(2n)x^{2n-1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}x^{2n-1}$$

Notice in the last step, our sum has changed so that it starts at $n = 1$. To see why, let's plug in $n = 0$ into our original sum. That term has $x^{2(0)} = x^0 = 1$, so it is constant. Thus when we take its derivative we will get 0, so we must omit the $n = 0$ term from our series, and so we start at $n = 1$.

Unfortunately, we don't recognize this series! All of the series we're used to start at $n = 0$. Let's **reindex** to get a sum that starts at 0. Let $k = n - 1$. This also means $k(1) = 1 - 1 = 0$, and $n = k + 1$, so our sum becomes

$$\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2(k+1)-1)!}x^{2(k+1)-1} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}x^{2k+1}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)(-1)^k}{(2k+1)!}x^{2k+1} = -\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = -\sin(x)$$

On the last line, we first factor out a -1. In the last step, notice that we have the sum for sine, except with k instead of n. The variable name we use for the index does not matter.

6 Solving linear 2^{nd} order ODEs using infinite series

Consider the linear differential equation

$$p(x)y^{''} + q(x)y^{'} + r(x)y = 0$$

where p, q, and r are polynomials. If x_0 is a point where $p(x_0) \neq 0$, we say that x_0 is an **ordinary point**. Otherwise it is a **singular point**. In this section, we look at a series solution at an ordinary (nonsingular) point.

6.0.6 Example: $y^{''} + y = 0$

Note that this is not quite the example from the book.

We try a power series solution centered at $x_0 = 0$. Note that our $p(x)$, the coefficient in front of $y^{''}$ is simply 1, so x_0 is an ordinary point. Start with the generic power series

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

We want to plug this series into our differential equation, but to do so we need $y^{''}$. So take derivatives:

$$y^{'} = \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$$

and

$$y^{''} = \frac{d}{dx}\sum_{n=1}^{\infty} n\, a_n x^{n-1} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

Notice that we have chosen to start the series for our derivatives at 1 and 2, respectively. As was pointed out in class, this is not necessary, because if you plug in $n = 0$ (and in the case of the 2^{nd} derivative, $n = 1$), those terms still become 0. However, it will matter in a moment when we add our sums together. In this problem, we will reindex our sums so that they all contain x^n. For this reindexing to work out, we'll need the sum for $y^{''}$ to start at $n = 2$.

Putting the series into our differential equation, we have

$$y^{''} + y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

Now let's reindex the first sum. Let $k = n - 2$ (and simultaneously, $n = k + 2$). So

$$y^{''} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$$

Let's change the index back to n, and plug into our differential equation:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n\, x^n = 0$$

$$= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + a_n\right]x^n = 0$$

For this to equal 0, it must equal 0 for all x, so

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

Thus

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}.$$

This is a recurrence relation/recursive definition which defines the sequence of numbers a_n. It says that if you know a number a_n, you can define the 2^{nd} number after it. Realize, however, that this means we don't know the first *two* numbers a_0 and a_1. The formula only starts helping us starting with a_2.

Let's figure out some values:

a_0	a_1
$a_2 = a_{(0)+2} = \dfrac{-a_0}{(0+2)(0+1)} = \dfrac{-a_0}{2!}$	$a_3 = a_{(1)+2} = \dfrac{-a_1}{(1+2)(1+1)} = \dfrac{-a_1}{3!}$
$a_4 = \dfrac{a_0}{(2+2)(2+1)} = \dfrac{a_0}{4\cdot 3\cdot 2!} = \dfrac{a_0}{4!}$	$a_5 = \dfrac{a_1}{5!}$
$\vdots$	$\vdots$
$a_{2n} = \dfrac{(-1)^n a_0}{(2n)!}$	$a_{2n+1} = \dfrac{(-1)^n a_1}{(2n+1)!}$

So the terms of our series have a factor of a_0 if they are even terms, and a factor of a_1 if they are odd terms. Let's plug in the a_n's into our original power series, but we must remember that it needs to be split into even and odd terms:

$$y = \sum_{n=0}^{\infty} a_{2n}x^{2n} + \sum_{n=0}^{\infty} a_{2n+1}x^{2n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n a_0}{(2n)!}x^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n a_1}{(2n+1)!}x^{2n+1}$$

$$= a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} + a_1\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

$$= a_0\cos(x) + a_1\sin(x)$$